A More General Method of Substructure Mode Synthesis for Dynamic Analysis

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A method of substructure mode synthesis for determining the dynamic characteristics of an undamped system in a specified frequency band is investigated. The motion of each substructure is represented by the three mode sets: inertia, selected normal, and residual modes. Inertia and residual mode sets are introduced to approximate the truncated lower and higher normal modes. Based on energy considerations, various combinations of modal approximations of each substructure can be used. By applying the zero-eigenvalue coupling technique to interface compatibility and equilibrium, an independent set of system equations is derived. Numerical results indicate that the proposed synthesis method is effective.

Introduction

MANY coupling procedures[1-13] have been proposed for solving the natural frequencies and normal modes of large undamped structure systems. In general, the system is divided into a number of simpler components or substructures. An analytical investigation or an exprimental test may be performed on each substructure. The substructure modes are obtained by specifying the interface as fixed[1,2] or free,[3,4] or by considering inertia and/or stiffness loadings.[5] In order to improve the accuracy some statically determined auxiliary modes also have been proposed.[6-8]

Hou[4] first used free-interface mode representation to determine the system dynamic properties for an intermediate frequency band. However, the truncation of partial modes used in free-interface methods may result in poor accuracy. To reduce this truncation error, Klosterman and McClelland[12] extended the concepts of MacNeal[6] and Rubin,[7] and introduced inertia restraint and residual flexibility matrices. They outlined a coupling procedure using a coordinate transformation to impose displacement compatibility and force equilibrium on the interface coordinates at the joints. This method appears to be especially effective for coupling two substructures where one is represented in modal coordinates and the other in physical coordinates. It is also noted that, in a different approach, Meirovitch and Hale[9,10] represent the motion of each substructure by using admissible functions to simplify computations.

This paper provides an improved means of formulating the equations of motion of a system. Based on energy considerations, various combinations of modal approximations of each substructure can be used to determine natural frequencies and modes in the specified frequency band.

Mode Representation for Substructures

A system is defined as a structure composed of N_t substructures connected at their joints. Considering component s with $N^{(s)}$ degrees of freedom at the system natural frequency λ, the equation of motion may be written as

$$m^{(s)}\ddot{x}^{(s)} + k^{(s)}x^{(s)} = f^{(s)} \tag{1}$$

where $m^{(s)}$ and $k^{(s)}$ are the mass and stiffness matrices, respectively, and $x^{(s)}$ the column matrix of displacement due to internal force $f^{(s)}$ at the joints connected with other substructures. For simplicity, all s superscripts will be omitted in all of the following derivations. By applying a set of substructure modes, the displacement matrix can be expressed as

$$x = \sum_{r=1}^{N} \phi_r p_r \tag{2}$$

where p_r is the generalized coordinate and ϕ_r the rth free-interface substructure mode obtained from the corresponding eigenvalue problem

$$m\ddot{x} + kx = 0 \tag{3}$$

In order to avoid too many symbols, the time-dependent and time-independent variables are denoted by the same symbol, such as vector x used in $x(t) = x\exp(i\omega t)$. Substituting Eq. (2) into Eq. (1) and applying the orthogonality conditions, the generalized coordinates can be expressed as

$$p_r = \frac{-\phi_r^T f}{\lambda^2 m_r [1-(\omega_r/\lambda)^2]} \tag{4}$$

or

$$p_r = \frac{\phi_r^T f}{k_r [1-(\lambda/\omega_r)^2]} \tag{5}$$

where the rth modal mass and stiffness matrices are defined by

$$m_r = \phi_r^T m \phi_r \tag{6}$$

$$k_r = \phi_r^T k \phi_r \tag{7}$$

$$\omega_r^2 = k_r/m_r$$

If the dynamic characteristics in some specified frequency band, i.e., $\lambda' \le \lambda \le \lambda''$, are of interest, the mode representation of the substructure displacement may be separated into three terms as in Ref. 12.

$$x = \sum_{r=1}^{\ell-1} \frac{(-\phi_r^T f)\phi_r}{\lambda^2 m_r [1-(\omega_r/\lambda)^2]} + \sum_{r=\ell}^{h} \phi_r p_r + \sum_{r=h+1}^{N} \frac{(\phi_r^T f)\phi_r}{k_r [1-(\lambda/\omega_r)^2]} \tag{8}$$

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where $\omega_\ell \leq \lambda$ and $\omega_h \geq \lambda$.

Since no force is applied at interior coordinates for a free-vibration system, i.e., $f_i = 0$, f can be expresed as

$$f = [\alpha_i | \alpha_j] \left\{ \begin{array}{c} f_i \\ \hline f_j \end{array} \right\} = \alpha_j f_j \tag{9}$$

where i,j are the identifiers of interior and joint coordinate sets, respectively, and α_i and α_j the associated transformation matrices. Therefore, Eq. (8) may be rearranged and approximated as

$$x = \sum_{r=1}^{\ell-1} \phi_r m_r^{-1} \phi_r^T \alpha_j (-f_j/\lambda^2) + \sum_{r=\ell}^{h} \phi_r p_r + \sum_{r=h+1}^{N} \phi_r k_r^{-1} \phi_r^T \alpha_j f_j \tag{10}$$

It is noted that the specified frequency band can be chosen slightly broader than the actual frequency band of interest such that the above equation remains to be a good approximation. Let Eq. (10) be expressed in matrix form as

$$\{x\} = [\beta]\{p\} \tag{11}$$

where

$$[\beta] = [\Psi \Phi \Theta] \quad \text{and} \quad \{p\} = \left\{ \begin{array}{c} p_I \\ p_N \\ p_R \end{array} \right\}$$

Ψ, Φ, and θ are the mode sets associated with inertia, selected normal, and residual modes, respectively, and p_I, p_N, and p_R are the corresponding generalized coordinates, defined as

$$\Psi = \left(\sum_{r=1}^{\ell-1} \phi_r m_r^{-1} \phi_r^T \right) \alpha_j \tag{12}$$

$$\Phi = [\phi_\ell, \phi_{\ell+1}, \ldots, \phi_h] \tag{13}$$

$$\Theta = \left(\sum_{r=h+1}^{N} \phi_r k_r^{-1} \phi_r^T \right) \alpha_j \tag{14}$$

and

$$p_I = -(f_j/\lambda^2) \tag{15}$$

$$p_N = \left\{ \begin{array}{c} p_\ell \\ p_{\ell+1} \\ \ddots \\ p_h \end{array} \right\} \tag{16}$$

$$p_R = f_j \tag{17}$$

From the orthogonality conditions, the following relationships of these mode sets can be obtained.

$$\Psi^T m \Phi = \Psi^T k \Phi = 0 \tag{18}$$

$$\Phi^T m \Theta = \Phi^T k \Theta = 0 \tag{19}$$

$$\Theta^T m \Psi = \Theta^T k \Psi = 0 \tag{20}$$

Let Ψ_j, Θ_j denote the submatrices which collect of all joint coordinates of Ψ and Θ at the joints; i.e.,

$$\Psi_j = \alpha_j^T \Psi \tag{21}$$

$$\Theta_j = \alpha_j^T \Theta \tag{22}$$

And it can be proved that

$$\Psi^T m \Psi = \Psi_j \tag{23}$$

$$\Theta^T k \Theta = \Theta_j \tag{24}$$

where Ψ_j and Θ_j are symmetrical matrices with the order of number of coordinates at the joints.

Formulation of System Equations

In formulating the system mass and stiffness matrices, the primary criterion is the preservation of the potential and kinetic energy relationships. For a structure composed of N_t substructures, the kinetic (K.E.) and potential energies (P.E.) are given by

$$\text{K.E.} = \sum_{s=1}^{N_t} T^{(s)} \tag{25}$$

$$\text{P.E.} = \sum_{s=1}^{N_t} V^{(s)} \tag{26}$$

where

$$T^{(s)} = \tfrac{1}{2} \dot{x}^{(s)T} m^{(s)} \dot{x}^{(s)} \tag{27}$$

$$V^{(s)} = \tfrac{1}{2} x^{(s)T} k^{(s)} x^{(s)} \tag{28}$$

To reduce the order of final system equations, the previous substructure mode representation is used. This will give a transformation to convert these energy expressions from an original physical coordinate set to a reduced generalized coordinate set. Based on energy considerations, two modal approximations of substructure are discussed.

Modal Approximation

Case A

This is the case where modal approximation is made by including the contribution from high-frequency kinetic energy and low-frequency potential energy. Substituting Eq. (11) into Eqs. (27) and (28), the kinetic and potential energies of substructure s can be written as

$$T^{(s)} = \tfrac{1}{2} p^{(s)T} \tilde{m}^{(s)} p^{(s)} \tag{29}$$

$$V^{(s)} = \tfrac{1}{2} p^{(s)T} \tilde{k}^{(s)} p^{(s)} \tag{30}$$

The generalized substructure mass and stiffness matrices are defined, respectively, by

$$\tilde{m}^{(s)} = \beta^{(s)T} m^{(s)} \beta^{(s)} = \begin{bmatrix} \Psi_j & 0 & 0 \\ 0 & m_N & 0 \\ 0 & 0 & (\Theta^T m \Theta) \end{bmatrix} \tag{31}$$

and

$$\tilde{k}^{(s)} = \beta^{(s)T} k^{(s)} \beta^{(s)} = \begin{bmatrix} (\Psi^T k \Psi) & 0 & 0 \\ 0 & k_N & 0 \\ 0 & 0 & \Theta_j \end{bmatrix} \tag{32}$$

where

$$m_N = \text{diag}[m_\ell, m_{\ell+1}, \ldots, m_h]$$

$$k_N = \text{diag}[k_\ell, k_{\ell+1}, \ldots, k_h]$$

Note that the terms include physical mass and stiffness matrices, i.e., $\Psi^T k \Psi$ and $\Theta^T m \Theta$, and can be calculated for substructures represented in the finite element model. However, they are usually not provided by experimental testing.

Case B

This is the case where modal approximation is made by neglecting the contribution from high-frequency kinetic energy and low-frequency potential energy. With a highly complex substructure, it is sometimes not economic or practical to apply finite element method to formulate and solve a dynamic model. In this case, it may be more efficient to obtain the required modal data from testing. However, there is a basic and inherent difficulty in attempting to derive a workable algorithm for synthesizing the mass and stiffness matrices from experimental data, because, in practice, the number of coordinates measured is usually larger than the number of eigenparameters determined. As a result, only a portion of eigenparameters is available for synthesizing the matrices. Several techniques have been proposed for overcoming this difficulty; however, no method has been generally accepted as a means of deriving these matrices.

In order to remove this difficulty, it is desired to have a different modal approximation to represent the system kinetic and potential energies from experimentally derived modal data.

In the range of the specified frequency band, the contribution of kinetic energy due to the high-frequency term $\Theta^T m \Theta$ may be negligible compared to that due to Ψ_j and m_N as experienced by the approximation of finite element method to a continuous system of infinite degrees of freedom. Furthermore, because of small strain energy attributed to low frequency, the contribution of potential energy due to the term $\Psi^T k \Psi$ may be negligible compared to that due to k_N and Θ.

Thus, the generalized substructure mass and stiffness matrices, $\tilde{m}$ and $\tilde{k}$, can be approximated as

$$\hat{m}^{(s)} = \begin{bmatrix} \Psi_j & 0 & 0 \\ 0 & m_N & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{33}$$

$$\hat{k}^{(s)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & k_N & 0 \\ 0 & 0 & \Theta_j \end{bmatrix} \tag{34}$$

One justification of this approximation is demonstrated in the examples by comparing the results with cases A and B as shown in Figs. 1-4. To simplify the expression in formulating the system equation, all of the substructures are considered in case A modal approximation such as obtained by the finite element method. The only change is that $\tilde{m}$ and $\tilde{k}$ will be replaced by $\hat{m}$ and $\hat{k}$ when the substructure is in case B modal approximation, which may be obtained by experiments. Let

$$M = \begin{bmatrix} \tilde{m}^{(1)} & & & \\ & \tilde{m}^{(2)} & & \\ & & \ddots & \\ & & & \tilde{m}^{(N_t)} \end{bmatrix} \tag{35a}$$

$$K = \begin{bmatrix} \tilde{k}^{(1)} & & & \\ & \tilde{k}^{(2)} & & \\ & & \ddots & \\ & & & \tilde{k}^{(N_t)} \end{bmatrix} \tag{35b}$$

$$P = \begin{Bmatrix} p^{(1)} \\ p^{(2)} \\ \vdots \\ p^{(N_t)} \end{Bmatrix} \tag{35c}$$

All elements in M and K otherwise designated are null. Thus, the kinetic and potential energies of the system can be written in matrix form.

$$\text{K.E.} = \tfrac{1}{2}\dot{P}^T M \dot{P} \tag{36}$$

$$\text{P.E.} = \tfrac{1}{2} P^T K P \tag{37}$$

The generalized coordinates P in Eqs. (36) and (37) are not all independent, but are related by displacement compatibility and force equilibrium at the joint coordinates of the substructures. These constraints can be expressed in generalized coordinates.

$$RP = 0 \tag{38}$$

Introducing the system Lagrangian defined as

$$L = \tfrac{1}{2}\dot{P}^T M \dot{P} - \tfrac{1}{2} P^T K P + \epsilon^T R P \tag{39}$$

ϵ is the Lagrangian multiplier vector.

The rth system equation of motion can be obtained by applying Lagrange's equation in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\xi}_r}\right) - \frac{\partial L}{\partial \xi_r} = \eta_r \tag{40}$$

where η_r is the generalized force associated with generalized coordinate ξ_r. For the substructure coupling problem, with forces exerted only at the interface of joining substructures, η_r must be zero which can be derived from the virtual work expression. Substituting Eq. (39) into Eq. (40) results in the system equation of motion

$$M\ddot{P} + KP = R^T \epsilon \tag{41}$$

Table 1 Percentage errors of frequencies including inertia and/or residual mode sets

Elastic mode No.	System frequency, Hz	Percentage errors, E_r, % Synthesis method[a] with N	$I+N$	$N+R$	$I+N+R$
5	352.34	−1.67	6.15	−7.05	0.00
6	492.95	−3.05	5.09	−7.44	0.00
7	657.98	0.60	4.93	−4.87	0.01
8	848.13	5.87	8.73	−2.15	0.01
9	1064.22	1.60	4.06	−3.63	0.10

[a] I = inertia mode set; N = selected normal mode set[4]; and R = residual mode set.

Table 2 Modal coherence values of modes including inertia and/or residual mode sets

Elastic mode No.	Modal coherence values C_{rs} Synthesis method with N	$I+N$	$N+R$	$I+N+R$
5	0.731	0.937	0.822	1.000
6	0.734	0.919	0.858	1.000
7	0.783	0.891	0.927	1.000
8	0.750	0.834	0.961	1.000
9	0.722	0.817	0.951	0.999

[a] I = inertia mode set; N = selected normal mode set[4]; and R = residual mode set.

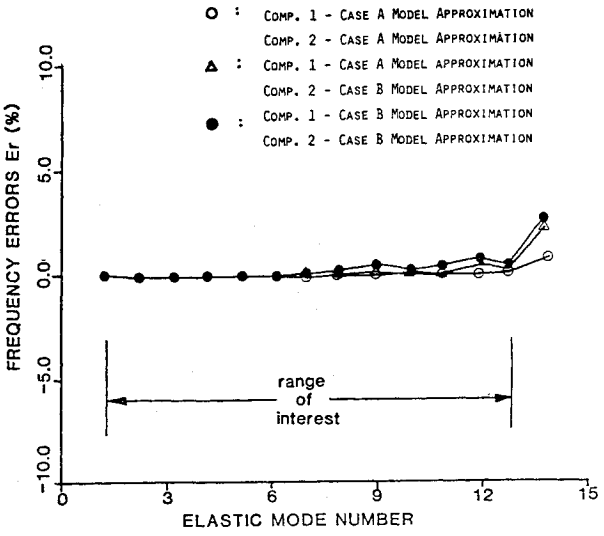

Fig. 1 Percentage errors of natural frequencies in 0-80 Hz.

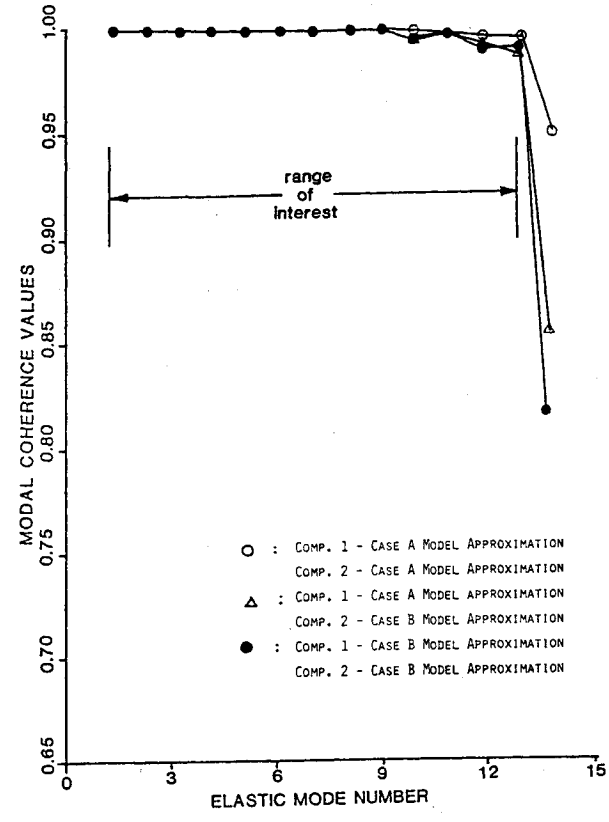

Fig. 2 Modal coherence values of computed modes in 0-80 Hz.

together with the constraint equation

$$RP=0 \tag{42}$$

Usually, an elimination algorithm is used to derive the system equations in an independent coordinate set. However, it may not be easy to identify the rank and the linearly independent columns of R. Here a zero-eigenvalue coupling technique proposed by Walton and Steeves[14] is applied to avoid this difficulty. Let a symmetric matrix D be defined as

$$D=R^TR \tag{43}$$

The rank of D may be equal to or less than the number of rows of R, depending on whether some redundant constraint equations may be presented in Eq. (42). Let N_r be the number of rows and N_c the number of columns of matrix R. Then the eigenvalue equation

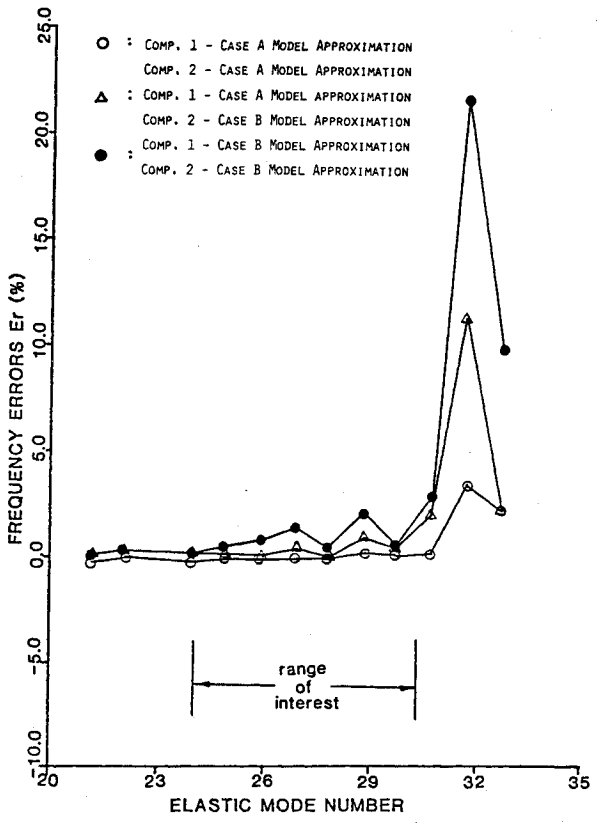

Fig. 3 Percentage errors of natural frequencies in 140-220 Hz.

$$DP=\lambda_d P \tag{44}$$

has at least (N_c-N_r) zero eigenvalue. Next let ζ_r (for $r=1,2,\ldots,N_q$) be the independent eigenvector associated with these zero eigenvalues, where $N_q>(N_c-N_r)$. Let β' be the $N_c\times N_q$ matrix whose columns are ζ_r. Thus,

$$D\beta'=R^TR\beta'=0 \tag{45}$$

Premultiplying Eq. (45) by β'^T, it is seen that

$$(R\beta')^T(R\beta')=0$$

so

$$R\beta'=0 \tag{46}$$

Therefore, β' is an orthogonal complement of R. The generalized coordinates P can be expressed in terms of N_q independent generalized coordinates Q through the relation

$$P=\beta'Q \tag{47}$$

Due to Eq. (46), the constraint equations in Eq. (42) are satisfied.

$$RP=R\beta'Q=0$$

By substituting Eq. (47) into Eq. (41) and premultiplying β'^T, the equation of motion is given in terms of an independent coordinate set Q.

$$M^*\ddot{Q}+K^*Q=0 \tag{48}$$

where $M^*=\beta'^TM\beta'$ and $K^*=\beta'^TK\beta'$.

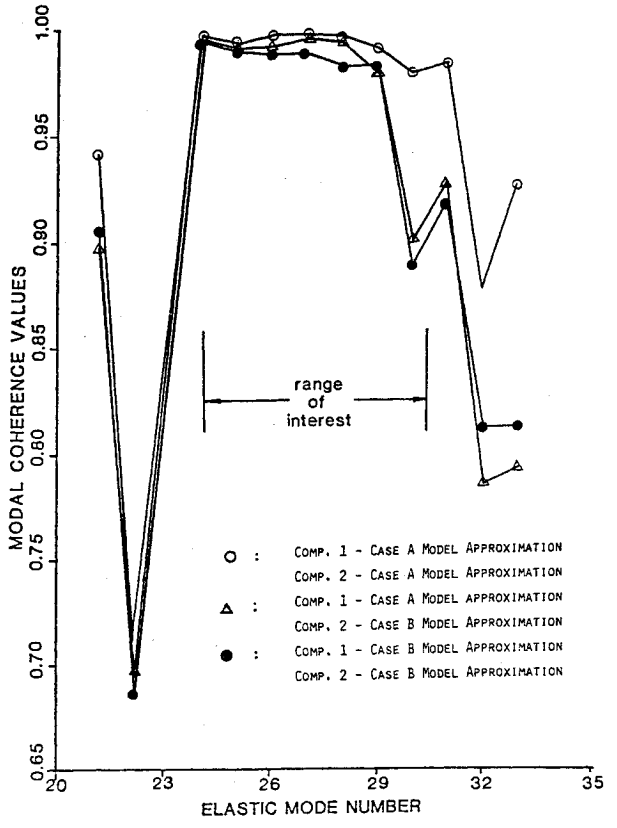

Fig. 4 Modal coherence values of computed modes in 140-220 Hz.

To demonstrate the synthesizing procedure, it is illustrated for a system having only two components. Let components 1 and 2 be represented in case A and case B modal approximation, respectively. The uncoupled system mass and stiffness matrices are constructed in the form

$$M=\begin{bmatrix} \tilde{m}^{(1)} & 0 \\ 0 & \hat{m}^{(2)} \end{bmatrix}; \quad K=\begin{bmatrix} \tilde{k}^{(1)} & 0 \\ 0 & \hat{k}^{(2)} \end{bmatrix} \tag{49}$$

The displacement compatibility of the joint coordinates, i.e., $x_j^{(1)}=x_j^{(2)}$, may be expressed in general coordinates as

$$\alpha_j^{(1)\mathrm{T}}\beta^{(1)}p^{(1)}=\alpha_j^{(2)\mathrm{T}}\beta^{(2)}p^{(2)} \tag{50}$$

Two additional sets of constraint equations derived from the definitions of p_I, p_R, and the equilibrium conditions at the joints can be written as

$$p_I^{(1)}+p_I^{(2)}=0 \tag{51}$$

$$p_R^{(1)}+p_R^{(2)}=0 \tag{52}$$

Thus, Eqs. (50-52) can be combined to give the constraint matrix R as

$$R=\begin{bmatrix} \Psi_j^{(1)} & \Phi_j^{(1)} & \Theta_j^{(1)} & -\Psi_j^{(2)} & -\Phi_j^{(2)} & -\Theta_j^{(2)} \\ I^{(1)} & 0 & 0 & I^{(2)} & 0 & 0 \\ 0 & 0 & I^{(1)} & 0 & 0 & I^{(2)} \end{bmatrix} \tag{53}$$

Next, the zero eigenvalue technique is employed to formulate the system equations in terms of an independent coordinate Q. The appropriate transformation can be used to convert the computed system eigenvectors from the reduced subspace to the original physical space.

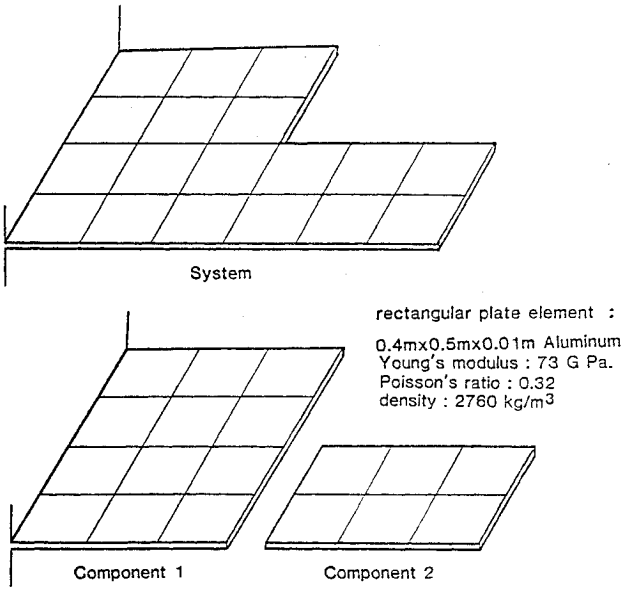

Fig. 5 Cantilever plate structure and components.

Numerical Example

Two error estimations have been introduced here to describe the relative errors of computed eigenvalues and eigenvectors. To evaluate the relative errors of the rth computed eigenvalues λ_r, they are compared to the corresponding eigenvalue λ_r^* solved by the finite element method on the entire system. Each of these errors is given by

$$E_r=(\lambda_r/\lambda_r^*-1)\times 100\% \tag{54}$$

where E_r is the percentage error of the rth computed eigenvalue. In comparing the accuracy of computed eigenvectors, a modal coherence is used as

$$C_{rs}=\frac{|\phi_r^T\phi_s^*|^2}{(\phi_r^T\phi_r)(\phi_s^{*T}\phi_s^*)} \tag{55}$$

where ϕ_r is the rth computed eigenvector and ϕ_s^* the sth eigenvector solved by the finite element method directly. The constant C_{rs} varies from a value of zero, representing no correlation, to one, representing a perfect consistent correspondence.

To demonstrate the effectiveness of this substructure mode synthesis method, a plate structure is chosen to be analyzed. The plate structure, as shown in Fig. 5, consists of two plate substructures with uniform stiffness and mass properties. Three degrees of freedom are defined for each node, consisting of a vertical translation and two rotations about the in-plane axes. The finite element model of the entire system has matrices of order 72. Two frequency bands are investigated. Case 1 is from 0 to 80 Hz, and case 2 is from 140 to 220 Hz. In case 1, 36 generalized coordinates, nine and ten selected normal modes for substructure 1 and 2 plus nine residual modes for each substructure, are employed in uncoupled system equations. The rank of the associated constraint matrix is 18. Hence, the coupled final system equation gives a reduction from an order of 72 to 18. The values of percentage error and modal coherence are plotted in Figs. 1 and 2. It is seen that all different combinations of mode approximation give good results.

In case 2, seven normal modes (modes 14-20) of substructure 1 and five normal modes (modes 8-12) of substructure 2 are selected. In addition to these selected normal modes, nine inertia modes and nine residual modes of each substructure

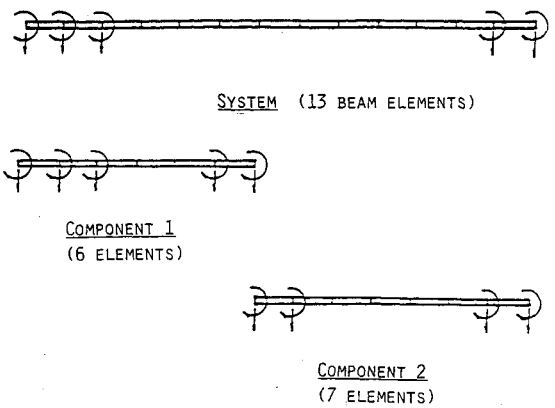

Fig. 6 Beam structure and subdivided states.

have to be included. The rank of constraint matrix is 27 in this case. A system equation of 21 independent generalized coordinates is formulated. The results are plotted in Figs. 3 and 4. Thus the order of the matrix from the whole structure by the finite element method is reduced from 72 to 21 using the method of the substructure mode synthesis.

In order to demonstrate the significance of including inertia and residual mode sets, a free-free bending beam system is analyzed by excluding (Hou's method) and including these mode sets. The system is shown in Fig. 6. A uniform rod is divided into two substructures; one consists of six elements and the other consists of seven. Two degrees of freedom are considered for every node point. The eigenparameters of this assembled beam structure located in the band of 300-1200 Hz are of interest. The results of different combinations of mode sets are tabulated in Tables 1 and 2.

Conclusions

1) A modal reduction technique is introduced to retain the most important information of each substructure in a specified frequency band.

2) Two modal approximations are employed in various combinations for coupling.

3) The zero-eigenvalue coupling procedure is applied to derive the equations of motion for the system in terms of independent generalized coordinates.

4) Numerical results indicate that the proposed synthesis method is effective.

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